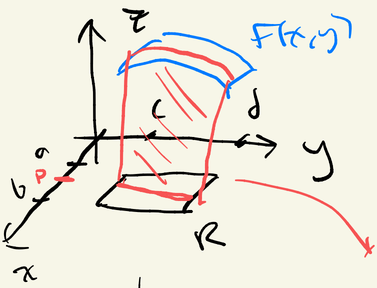
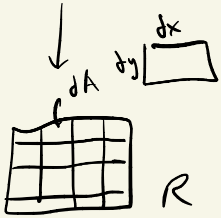
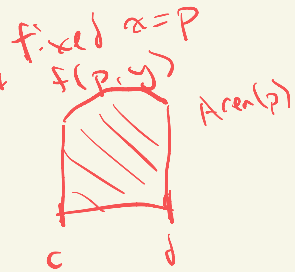
Aug 4

Last time: Double and iterated integrals



$$\text{Volume} = \int_a^b \underbrace{\text{Area}(x)}\, dx$$

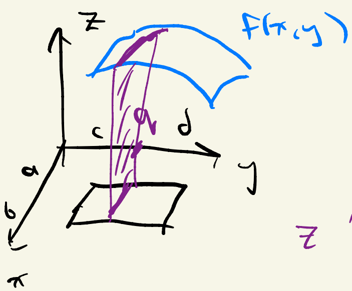




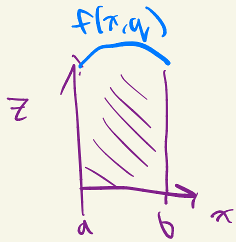
$$\text{Area}(p) = \int_c^d f(p,y)\, dy$$

since p is fixed, f(x,y) is now a 1 variable function

Fix y=q



$$\text{Vol} = \int_c^d \underbrace{\text{Area}(y)}\, dy$$

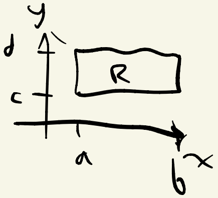


$$\text{Area}(q) = \int_a^b f(x,q)\, dx$$

$\Rightarrow$ Fubini's Thm

Thm 1 Fubini's Thm (First form):

If $f(x,y)$ is continuous in rectangle R, $a \le x \le b$, $c \le y \le d$



Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

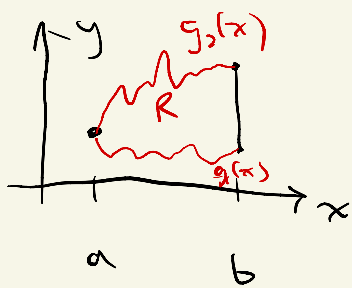
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Thm 2: Stronger Fubini

$f(x,y)$ continuous on R

- Region R is $a \le x \le b$

  $g_1(x) \le y \le g_2(x)$



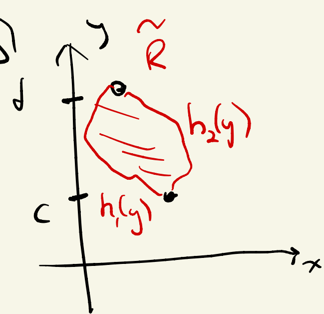
with g_1, g_2 continuous on a, b

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

- Similarly: Region $\tilde{R}$ $c \le y \le d$

  $h_1(y) \le x \le h_2(y)$

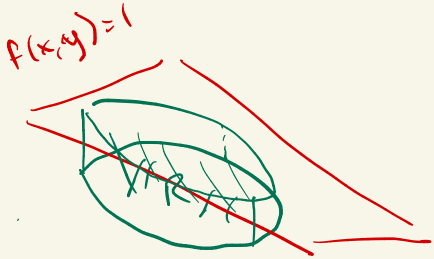


$$\iint_{\tilde{}} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Today 15.3 (area and average value)

Recall from extra credit tuesday:

we can use double integrals to find areas

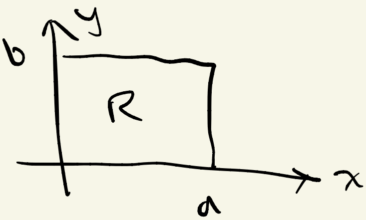


$$\underset{[m^3]}{\text{Vol}} = \iint_R f(x,y)\, dA$$

$$= \iint_R 1 \; dA$$

$$= \underset{[m^2]}{\text{Area}(R)} \cdot \underset{[m]}{1}$$

Ex:



$$\iint_R 1 \, dx\, dy = \int_0^b \int_0^a dx\, dy$$

$$= \int_0^b a \, dy$$

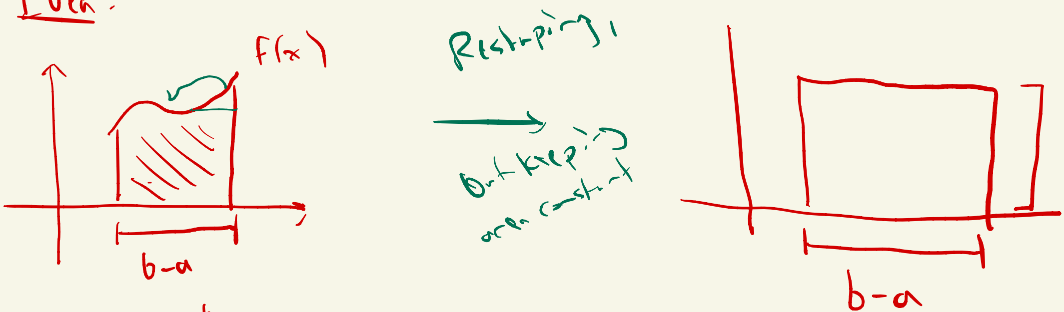
length·width $\rightarrow$ $= ab$

Recall: Average Value 21B

Average value Ave(f) of a function f(x) on [a, b]

$$\text{Ave}(f) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Idea:



$$\int_a^b f(x)\,dx = (b-a)\cdot \text{Ave}(f)$$

Def:

The Average Value Ave(f) of a function f(x, y) on a region $R \subseteq \mathbb{R}^2$ is

$$\text{Ave}(f) = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$$

Ex: Average power delivered by light to a solar cell

$$f(x,y) = 2 + \sin(x+y)$$

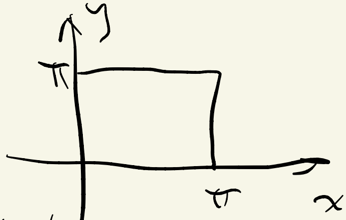
over rectangle

$0 \le x, y \le \pi$

$0 \le x \le \pi$

$0 \le y \le \pi$

$$f(x,y) = 2 + \sin(x+y)$$



$$\text{Area}(R) = \pi^2$$

$$\text{Ave}(f) = \frac{1}{\text{Area}(R)} \iint_R 2 + \sin(x+y)\,dA$$

$$= \frac{1}{\pi^2} \int_0^{\pi}\int_0^{\pi} \underline{2} + \underline{\sin(x+y)}\,dx\,dy$$

constant + variable

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} 2\,dx\,dy + \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(x+y)\,dx\,dy$$

$$= \frac{2\pi^2}{\pi^2} + \frac{1}{\pi^2}\int_0^{\pi}\Big[-\cos(x+y)\Big]_{x=0}^{x=\pi} dy$$

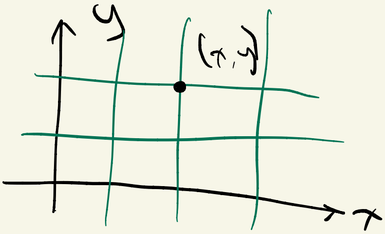
$$= 2 + \frac{1}{\pi^2}\int_0^{\pi} -\cos(y+\pi) + \cos(y)\,dy$$

$$= 2 + \frac{1}{\pi^2}\Big[\overbrace{-\sin(y+\pi) + \sin(y)}^{0}\Big]_0^{\pi}$$

$$= 2$$

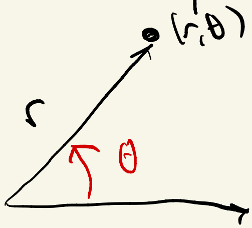
15.4 Polar Coordinates

Rectangular coordinates



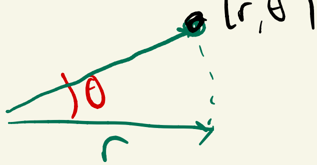
// Start at $(0,0)$. Move right x,
move up y
(or move up y, move right x)

Another way: Start staring at $x = +\infty$



- Turn Lhand θ radians
- walk forward r distance

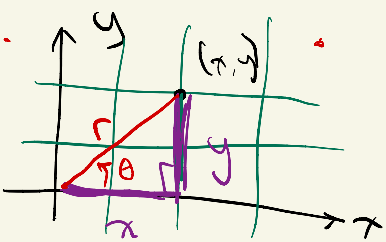
(note: should also work for walk forward r distance, and rotate θ radians



Every point has an (r,θ):

$r \geq 0$ (it's a distance)

$0 \leq \theta \leq 2\pi$



$\frac{y}{x} = \tan(\theta)$

$\theta = \arctan\left(\frac{y}{x}\right)$

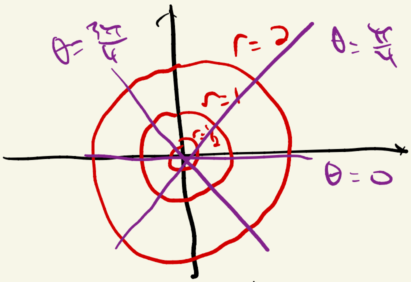
From trig:

$\Rightarrow$
$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}}$$
(pythagorean)

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

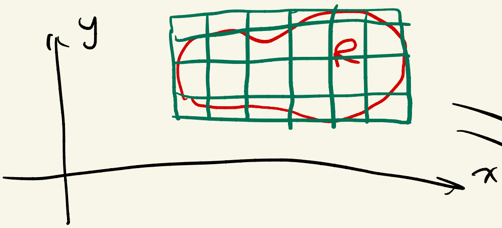
"Grid" in polar:

r are distance to origin



How to integrate?

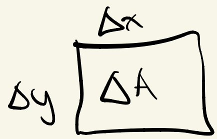
- Rectangular:



in limit of tiny rectangles

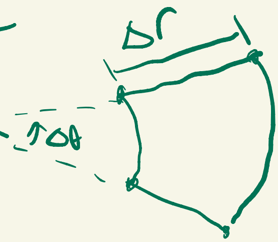
$$\iint_R f\, dA \approx \sum_{\text{rectangles}} f(x,y)\,\Delta A$$

$$= \sum_{\text{rectangles}} f(x,y)\,\Delta x \Delta y$$



$\Delta A = \Delta x \Delta y$

- Polar:

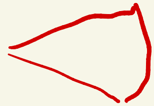
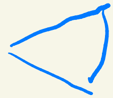


little areas are wedges

$$\iint_R f\, dA \approx \sum_{\text{wedges}} f(r,\theta)\,\Delta A$$

What is ΔA?

How to calculate wedge area:

ΔA  $=$  $-$ 

$$= \frac{(r+\Delta r)^2\Delta\theta}{2} - \frac{r^2\Delta\theta}{2}$$

$$= \left((r+\Delta r)^2 - r^2\right)\frac{\Delta\theta}{2}$$

$$= \left(r^2 + 2r\Delta r + (\Delta r)^2 - r^2\right)\frac{\Delta\theta}{2}$$

$$= \left(2r\Delta r + (\Delta r)^2\right)\frac{\Delta\theta}{2}$$

$$\approx (2r\Delta r)\frac{\Delta\theta}{2}$$

$$\boxed{= r\Delta r\Delta\theta}$$

(Think: circles: $\Delta\theta = 2\pi$,
Area $= \pi r^2$
$= \frac{\boxed{2\pi} r^2}{2}$
Area actual: $\frac{\Delta\theta r^2}{2}$

initially, $\cancel{(\Delta r)^2}$ too small to matter

we care about limit $\Delta r \to 0$
$(\Delta r)^2 \to 0$ much faster: →
$\lim_{\Delta r\to 0} \frac{(\Delta r)^2}{\Delta r} = \lim_{x\to 0} x^2/x = \lim_{x\to 0} x = 0$

$\Delta A = r\Delta r\Delta\theta + (\text{something tiny})$

$\boxed{11:00}$

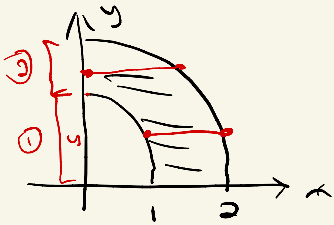
$\Rightarrow \quad dA = r\,dr\,d\theta$

$$\Rightarrow \iint_R f\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta$$ ← note this!

Ex:

Find $\iint_R x^2+y^2\,dA$ where R is quarter annulus $1\le r\le 2$ is first quadrant

Soln:



In rectangular:

Fix y

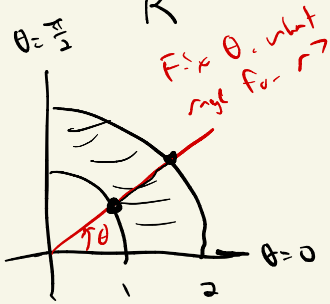
$$\iint_R x^2+y^2\,dA = \int_{y=0}^{1}\int_{x=\sqrt{1-y^2}}^{x=\sqrt{4-y^2}} x^2+y^2\,dx\,dy \quad ①$$

$$+ \int_{y=1}^{y=2}\int_{x=0}^{x=\sqrt{4-y^2}} x^2+y^2\,dx\,dy \quad ②$$

yikes!

But observe:

$$\iint_R x^2+y^2\,dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=1}^{r=2} \underbrace{r^2}\, r\,dr\,d\theta$$



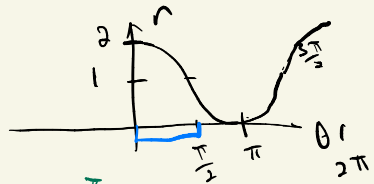
$$= \int_0^{\frac{\pi}{2}}\int_1^2 r^3\,dr\,d\theta = 15\pi/8$$

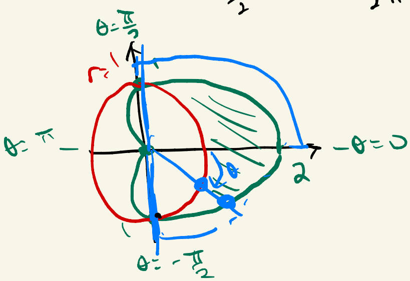
$$x^2+y^2 = (r\cos\theta)^2+(r\sin\theta)^2$$
$$= r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta+\sin^2\theta)$$
$$= r^2$$

Ex:

Find the area outside the unit circle centered at origin and inside cardioid $r = 1+\cos\theta$

Soln:





Superimpose

Approach: Fix θ. How does r vary?

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=1+\cos\theta} r \, dr \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_1^{1+\cos\theta} r \, dr \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left[\frac{1}{2}r^2\right]_{r=1}^{r=1+\cos\theta} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2\theta + 2\cos\theta \, d\theta$$

$$= \frac{\pi}{4} + 2$$

Ex:

Evaluate integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} x^2+y^2 \, dy \, dx$$

Soln:

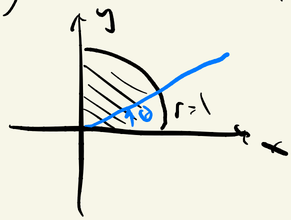
$\sqrt{1-x^2}$ suggests polar:

$$y = \sqrt{1-x^2}$$

$$\Rightarrow y^2 = 1 - x^2$$

$$1 = x^2+y^2$$

$\Rightarrow$ Since $x \geq 0$, $y \geq 0$,



Rectangular:

$$= \int_0^1 x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \, dx$$

nope.

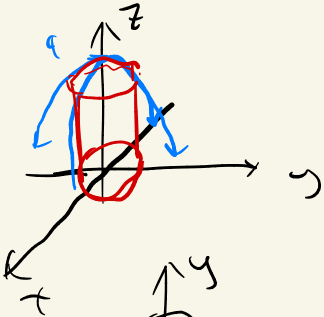
Instead: polar

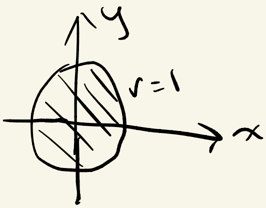
Fix θ. Then

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} r^2 \; r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \, dr\,d\theta$$

Ex:

Find volume of solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by unit circle in xy plane





$$\iint_R 9 - x^2 - y^2 \, dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (9 - r^2)\, r \, dr \, d\theta$$

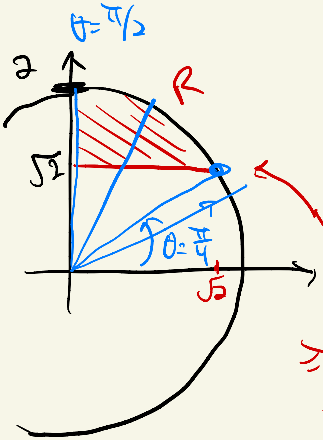
$-(x^2+y^2)$

$$= \int_0^{2\pi} \int_0^1 (9 - r^2)\, r \, dr \, d\theta$$

$$= \frac{17\pi}{2}$$

Ex:

Fix θ.



$y = \sqrt{2}$

$\Rightarrow r \sin\theta = \sqrt{2}$

$\Rightarrow r = \frac{\sqrt{2}}{\sin\theta}$

$$\iint_R 1 \, dA = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=\frac{\sqrt{2}}{\sin\theta}}^{r=2} 1 \cdot r \, dr \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2}\csc\theta}^{2} r \, dr \, d\theta$$

$$= \frac{1}{2}(\pi - 2)$$